"Niels Bohr (not for the first time) was ready to abandon the law of conservation of energy. It is interesting to note that Bohr was an outspoken critic of Einstein's light quantum (prior to 1924), that he discouraged Dirac's work on the relativistic electron theory (telling him, incorrectly, that Klein and Gordon had already succeeded), that he opposed Pauli's introduction of the neutrino, that he ridiculed Yukawa's theory of the meson, and that he disparaged Feynman's approach to quantum electrodynamics."

- Prof. David Griffiths, Reed College, excerpted from *Introduction to Elementary Particles*

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

An interesting (to me) point concerning the physical meaning of orthogonal polarization was raised after discussion section the other day. Fowles says that two waves $\mathbf{E}_1$ and $\mathbf{E}_2$ whose complex electric field amplitudes satisfy:

$$\mathbf{E}_1 \cdot \mathbf{E}_2^* = 0 \tag{1}$$

are orthogonally polarized.

For linear polarization, there is a simple geometric analogy. Linearly polarized light (e.g., in the $\hat{x}$ direction) is orthogonal to light with a perpendicular linear polarization (e.g., in the $\hat{y}$ direction). So no light will get through two linear polarizers which are "orthogonal" in the Euclidean geometry sense. However, this picture breaks down for more complicated polarization states, e.g. circular polarizations. For example, two circular polarizations whose electric fields are always at right angles to each other are *not* orthogonal!

Orthogonality for polarization states can be understood using notions from linear algebra. In this sense the complex vector space of polarization states can be spanned by two linearly independent, complex vectors - any two linearly independent complex vectors are said to be orthogonal. This is what Fowles means when he says two polarization states are orthogonal.

Another interesting question raised after discussion section, although it is a bit beyond the scope of the course, was whether or not a free-falling (in a gravitational field) charged particle radiates. You would expect that it might not based on the equivalence principle, which basically states that a free-falling frame is equivalent to an inertial frame. However, if you observe the charge from the surface of some planet, you would see a charged particle undergoing acceleration. This is a little funny, since you would expect that the charge either loses energy or it doesn't...

After thinking about it a little and consulting some wise general relativity texts (e.g., Wald's *General Relativity* or Misner, Thorne and Wheeler's *Gravitation*), I believe that the answer is frame dependent (as is the case for most relativity paradoxes). If you're free-falling with the particle, you don't see any radiation. If the charge is accelerating with respect to you, then you see radiation. This can be shown with the general relativistic field transformations. What about energy conservation? Well, I'm no Niels Bohr, so I think that if you change your acceleration into the frame of the particle, everything works out... but to prove this seems a bit complicated... anyhow, good stuff to think about, keep up the great work! Thanks!

Problem 1

First we'll calculate the electromagnetic energy radiated $P_{rad}\Delta t$ during the deceleration lasting for $\Delta t = v/a$, which is given by:

$$P_{rad}\Delta t = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2 av}{c^3}. \tag{2}$$

The electron's initial kinetic energy K is just:

$$K = \frac{1}{2}mv^2, \tag{3}$$

and the ratio $P_{rad}\Delta t/K$ is given by:

$$\frac{P_{rad}\Delta t}{K} = \frac{4}{3}\left(\frac{1}{4\pi\epsilon_0}\frac{e^2}{mc^2}\right)\left(\frac{a}{cv}\right). \tag{4}$$

The distance d traveled by light in Δt is cv/a, and the classical radius of the electron r_0 is given by the formula in the problem set: $r_0 \equiv e^2/\left(4\pi\epsilon_0 mc^2\right)$. Therefore:

$$\frac{P_{rad}\Delta t}{K} = \frac{4}{3}\left(\frac{r_0}{d}\right). \tag{5}$$

Here's an interesting fact that might help you remember some important lengths in physics:

One of the most important constants in physics is the fine-structure constant α, which sets the strength scale for the electromagnetic force:

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}, \tag{6}$$

where e is in CGS units. You may recognize the upper division physics course, Physics α^{-1}.

You can guess the the classical radius of the electron by setting the rest energy of the electron equal to the potential energy stored in a spherical shell of radius r_0 with charge e on the surface.

$$mc^2 = \frac{e^2}{r_0}. \tag{7}$$

In a few weeks, you'll learn about Compton scattering (photon-electron scattering). The Compton wavelength of the electron is given by r_0/α. The Bohr radius, the radius of an electron's orbit in the hydrogen atom, is given by r_0/α^2.

Problem 2

The electron oscillates sinusoidally with the acceleration given by:

$$a = \frac{|e|E_0}{m}\sin\omega t. \tag{8}$$

The power P_{rad} radiated is given by:

$$P_{rad} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^4}{c^3}\frac{E_0^2}{m^2}\sin^2\omega t, \tag{9}$$

and of course if we average over many cycles...

$$\langle P_{rad}\rangle = \frac{1}{4\pi\epsilon_0}\frac{1}{3}\frac{e^4}{c^3}\frac{E_0^2}{m^2}. \tag{10}$$

Next if we divide this result by the average power density $\langle U\rangle$ in the incident wave we get the scattering cross section σ:

$$\sigma = \frac{1}{6\pi\epsilon_0^2}\frac{e^4}{m^2c^4}. \tag{11}$$

You might notice that $\sigma = (8/3)\pi r_0^2$, r_0 being the classical electron radius from problem 1...

Problem 3

We transform to a comoving inertial frame F' in which the electron is temporarily at rest. The electric and magnetic fields in F' are given by:

$$\begin{aligned}\vec{E}'_\perp &= \gamma\left(\vec{E}_\perp + c\vec{\beta}\times\vec{B}\right)\\ \vec{B}'_\perp &= \gamma\left(\vec{B}_\perp - \frac{1}{c}\vec{\beta}\times\vec{E}\right)\\ \vec{E}'_{\parallel} &= \vec{E}_{\parallel}\\ \vec{B}'_{\parallel} &= \vec{B}_{\parallel}\\ &.\end{aligned} \tag{12}$$

The force on the electron is the Lorentz force given by:

$$\vec{F} = e\left(\vec{E} + \vec{v}\times\vec{B}\right), \tag{13}$$

but in the comoving frame F' the electron's velocity is zero. Since in the lab frame F the electric field is zero, the force acting on the electron in F' is:

$$F = eE'_\perp = e\gamma c\beta B. \tag{14}$$

Thus, the acceleration a is:

$$a = \frac{e\gamma c\beta B}{m} \tag{15}$$

This acceleration can be used in our old pal which describes the power radiated:

$$P'_{rad} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^2a^2}{c^3}, \tag{16}$$

which gives us:

$$P'_{rad} = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{e^4\gamma^2\beta^2B^2}{m^2c}. \tag{17}$$

When we transform back to frame F, the energy transforms as $\Delta E' \to \gamma\Delta E$ and the time transforms as $\Delta t' \to \gamma\Delta t$. Therefore, since power is just $\Delta E/\Delta t$, $P'_{rad} = P_{rad}$.

Problem 4

Fowles 2.4

For this problem we employ the complex exponential form of the wave functions for $\vec{E}$ and $\vec{H}$:

$$\vec{E} = Re\Big(\vec{E_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Big)$$
$$\vec{H} = Re\Big(\vec{H_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Big). \tag{18}$$

The Poynting vector is given by:

$$\vec{S} = \vec{E} \times \vec{H}. \tag{19}$$

Using the expressions from Eq. (18), we find the Poynting vector is:

$$\vec{S} = Re\Big(\vec{E_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Big) \times Re\Big(\vec{H_0}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Big) \tag{20}$$

We can expand the exponentials in terms of sines and cosines and find the real parts:

$$\vec{S} = \Big(Re(\vec{E_0})\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) - Im(\vec{E_0})\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)\Big)$$
$$\times\Big(Re(\vec{H_0})\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) - Im(\vec{E_0})\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)\Big). \tag{21}$$

If we expand this expression and time average (i.e. we set $\sin^2(\mathbf{k}\cdot\mathbf{r}-\omega t)$ and $\cos^2(\mathbf{k}\cdot\mathbf{r}-\omega t)$ equal to $\frac{1}{2}$ and $\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)$ equal to 0), then we get:

$$\langle\vec{S}\rangle = \frac{1}{2}\Big(Re(\vec{E_0}) \times Re(\vec{H_0}) + Im(\vec{E_0}) \times Im(\vec{H_0})\Big). \tag{22}$$

This expression, by inspection, is equivalent to:

$$\langle\vec{S}\rangle = \frac{1}{2}Re\Big(\vec{E_0} \times \vec{H_0^*}\Big), \tag{23}$$

which verifies the claim.

Problem 5

Fowles 2.7

This is pretty straightforward. Here's the prescription:

Given the electric field of the wave, $\vec{E}$, in the form:

$$\vec{E} = E_0\Big(\hat{i} + \hat{j}be^{i\theta}\Big)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{24}$$

the Jones vector is given by:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = E_0\begin{bmatrix} 1 \\ be^{i\theta} \end{bmatrix}. \tag{25}$$

You can normalize the Jones vector if you want, but if you didn't feel like doing that in this problem that's okay too. So pretty much we can just write down the answers, here they are:

(a)

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \sqrt{2}E_0\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{26}$$

(b)

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \sqrt{5}E_0\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}. \tag{27}$$

(c)

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \sqrt{2}E_0\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix}. \tag{28}$$

(d)

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = 2E_0\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{1+i}{2} \end{bmatrix}. \tag{29}$$

Problem 6

Fowles 2.10

We'll start with an arbitrary polarization:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} a \\ be^{i\theta} \end{bmatrix}. \tag{30}$$

Now we'll send it through a linear polarizer. Let's orient the linear polarizer at 45^o, so our resultant polarization is given by:

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ be^{i\theta} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a+be^{i\theta} \\ a+be^{i\theta} \end{bmatrix} = \frac{a+be^{i\theta}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{31}$$

We can ignore the amplitude out front. Now we'll sent it through a quarter-wave plate with the fast axis horizontal.

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}, \tag{32}$$

which is indeed circular polarization! What happens if we change the order?

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} a \\ be^{i\theta} \end{bmatrix} = \frac{a+be^{i\theta}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{33}$$

which is linear polarization. So circular polarized light is created only by placing the optical elements in the proper order.

Problem 7

(a)

For this problem, we can use the technique for changing the basis of a matrix operator that we employed in the first problem set to find Lorentz transforms in rotated frames (see PS1 solutions, problem 2). Namely,

$$\mathcal{M}' = R^{-1}\mathcal{M}R. \tag{34}$$

where $\mathcal{M}$ is the Jones matrix and R is the appropriate rotation matrix. For a linear polarizer with transmission axis at an arbitrary angle ϕ:

$$\mathcal{M} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}. \tag{35}$$

(b)

A quick way to check if the matrix is unitary is to multiply $\mathcal{M}$ by $\left(\mathcal{M}^T\right)^*$ and see if it equals the identity matrix:

$$\begin{aligned} \mathcal{M}\cdot\left(\mathcal{M}^T\right)^* &= \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}\begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}. \end{aligned} \tag{36}$$

Well that's not the identity matrix, so the Jones matrix of a linear polarizer is not unitary.

Problem 8

(a)

The Jones matrix for the ideal wave plate is:

$$\mathcal{M} = \begin{bmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{bmatrix}. \tag{37}$$

(b)

For the general case we just do the matrix multiplication as in problem 7 (a):

$$\begin{aligned} \mathcal{M}' &= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \\ &= e^{-i\delta/2}\begin{bmatrix} e^{i\delta}\cos^2\phi - \sin^2\phi & \left(1+e^{i\delta}\right)\sin\phi\cos\phi \\ -\left(1+e^{i\delta}\right)\sin\phi\cos\phi & \cos^2\phi - e^{i\delta}\sin^2\phi \end{bmatrix}. \end{aligned} \tag{38}$$

(c)

Now we check for unitarity just as in problem 7 (b):

$$\begin{aligned} \mathcal{M}'\cdot\left(\mathcal{M}'\right)^{T*} &= e^{-i\delta/2}\begin{bmatrix} e^{i\delta}\cos^2\phi - \sin^2\phi & \left(1+e^{i\delta}\right)\sin\phi\cos\phi \\ -\left(1+e^{i\delta}\right)\sin\phi\cos\phi & \cos^2\phi - e^{i\delta}\sin^2\phi \end{bmatrix}. \\ & e^{i\delta/2}\begin{bmatrix} e^{-i\delta}\cos^2\phi - \sin^2\phi & \left(1+e^{-i\delta}\right)\sin\phi\cos\phi \\ -\left(1+e^{-i\delta}\right)\sin\phi\cos\phi & \cos^2\phi - e^{-i\delta}\sin^2\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{39}$$

So the Jones vector for the waveplate is a unitary operator!

Bye for now!